

GLOBAL ASYMPTOTICS OF STIELTJES-WIGERT POLYNOMIALS

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ABSTRACT. Asymptotic formulas are derived for the Stieltjes-Wigert polynomials $S_n(z; q)$ in the complex plane as the degree n grows to infinity. One formula holds in any disc centered at the origin, and the other holds outside any smaller disc centered at the origin; the two regions together cover the whole plane. In each region, the q -Airy function $A_q(z)$ is used as the approximant. For real $x > 1/4$, a limiting relation is also established between the q -Airy function $A_q(x)$ and the ordinary Airy function $\text{Ai}(x)$ as $q \rightarrow 1$.

1. INTRODUCTION

We first fix some notations. Let $k > 0$ be a fixed number and

$$(1) \quad q = \exp\{-(2k^2)^{-1}\}.$$

Note that $0 < q < 1$. The q -shifted factorial is given by

$$(a, q)_0 = 1, \quad (a, q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad n = 1, 2, \dots.$$

With these notations, the Stieltjes-Wigert polynomials

$$(2) \quad S_n(x; q) = \sum_{j=0}^n \frac{q^{j^2}}{(q, q)_j (q; q)_{n-j}} (-x)^j, \quad n = 0, 1, 2, \dots,$$

are orthogonal with respect to the weight function

$$(3) \quad w(x) = k\pi^{-\frac{1}{2}} \exp\{-k^2 \log^2 x\}$$

for $0 < x < \infty$; see [9, (18.27.18)] and [8, (3.27.1)]. It should be mentioned that the Stieltjes-Wigert polynomials belong to the indetermined moment class and the weight function in (3) is not unique; see [3]. One important property of the Stieltjes-Wigert polynomials is the symmetry relation

$$(4) \quad S_n(z; q) = (-zq^n)^n S_n\left(\frac{1}{zq^{2n}}; q\right),$$

which can be easily verified by changing the index j to $n-j$ in the explicit expression given in (2). In some literatures, the variable x in (2) is replaced by $q^{\frac{1}{2}}x$; see, for example, Szegő [11], Chihara [2], and Wang and Wong [14]. The notation for the Stieltjes-Wigert polynomials used in these literatures is

$$(5) \quad p_n(x) = \frac{(-1)^n q^{n/2+1/4}}{\sqrt{(q; q)_n}} S_n(q^{\frac{1}{2}}x; q).$$

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The Stieltjes-Wigert polynomials appear in random walks and random matrix formulation of Chern-Simons theory on Seifert manifolds; see [1, 4].

The asymptotics of the Stieltjes-Wigert polynomials, as the degree tends to infinity, has been studied by several authors. In 1923, Wigert [15] proved that the polynomials have the limiting behavior

$$(6) \quad \lim_{n \rightarrow \infty} (-1)^n q^{-n/2} p_n(x) = \frac{q^{1/4}}{\sqrt{(q; q)_\infty}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2+k/2}}{(q; q)_k} x^k,$$

which can be put in terms of the q -Airy function

$$(7) \quad A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} (-z)^k.$$

This function appeared in the third identity on p.57 of Ramanujan's "Lost Notebook" [10]. (For this reason, it is also known as the Ramanujan function.) In terms of the q -Airy function, Wigert's result can be stated as

$$(8) \quad \lim_{n \rightarrow \infty} S_n(x; q) = \frac{1}{(q; q)_\infty} A_q(x).$$

It is known that all zeros of $S_n(x; q)$ lie in the interval $(0, 4q^{-2n})$; see [14]. Hence, we introduce a new scale

$$(9) \quad z := q^{-nt} u$$

with $u \in \mathbb{C} \setminus \{0\}$ and $t \in \mathbb{R}$. The values of $t = 0$ and $t = 2$ can be regarded as the turning points of $S_n(q^{-nt}u; q)$. Taking into account the symmetry relation in (4), one may restrict oneself to the case $t \geq 1$; see [13, (1.4)]. (However, in the present paper, we will not make this restriction.) The case $t = 2$ has been studied by Ismail [6], and he proved

$$(10) \quad \lim_{n \rightarrow \infty} q^{n^2} (-u)^n S_n(uq^{-nt}; q) = \frac{1}{(q; q)_\infty} A_q\left(\frac{1}{u}\right), \quad t = 2,$$

uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$; see [6, Theorem 2.5]. This result can in fact be derived directly from Wigert's result in (8) via the symmetry relation mentioned in (4). In [7], Ismail and Zhang extended the validity of this result to $t \geq 2$. For $1 \leq t < 2$, Ismail and Zhang [7] gave asymptotic formulas for these polynomials in terms of the theta-type function

$$(11) \quad \Theta_q(z) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k,$$

but in a very complicated manner. The result in [7] was then simplified by Wang and Wong [12]. For instance, when $1 \leq t < 2$, Wang and Wong proved that

$$(12) \quad S_n(uq^{-nt}; q) = \frac{(-u)^{n-m} q^{n^2(1-t)-m[n(2-t)-m]}}{(q; q)_n (q; q)_\infty} \left\{ \Theta_q\left(\frac{q^{2m-n(2-t)}}{-u}\right) + \mathcal{O}(q^{n(l-\delta)}) \right\},$$

where $l = \frac{1}{2}(2-t)$, $m = \lfloor nl \rfloor$ and $\delta > 0$ is any small number; see [12, Corollary 2]. Note that all these results are not valid in a neighborhood containing $t = 2$, one of the turning points. To resolve this issue, a uniform asymptotic formula was given by Wang and Wong in

a second paper [13]. For $z := uq^{-nt}$ with $t > 2(1 - \delta)$, δ being any small positive constant, they showed that

$$(13) \quad S_n(z; q) = \frac{(-z)^n q^{n^2}}{(q; q)_n} [A_{q,n}(q^{-2n}/z) + r_n(z)],$$

where $r_n(z)$ is the remainder and $A_{q,n}(z)$ is the q -Airy polynomial obtained by truncating the infinite series in (7) at $k = n$, i.e.,

$$A_{q,n}(z) = \sum_{k=0}^n \frac{q^{k^2}}{(q, q)_k} (-z)^k.$$

In this paper, we shall show that the q -Airy polynomial $A_{q,n}(z)$ in equation (13) can be replaced by the q -Airy function $A_q(z)$. Moreover, we shall show that the resulting formula is global. More precisely, we have the following result.

Theorem 1. *Let $z := uq^{-nt}$ with $-\infty < t < 2$, $u \in \mathbb{C}$ and $|u| \leq R$, where $R > 0$ is any fixed positive number. We have*

$$(14) \quad S_n(z; q) = \frac{1}{(q; q)_n} [A_q(z) + r_n(z)],$$

where the remainder satisfies

$$(15) \quad |r_n(z)| \leq \left[\frac{q^{n(1-\sigma)}}{1-q} + \frac{2}{1-q} \left(\frac{1}{2} \right)^{\lfloor n\sigma \rfloor} \right] A_q(-|z|)$$

with $\sigma = \max\{\frac{1}{2}, \frac{1}{2} + \frac{t}{4}\}$.

Let $z := uq^{-nt}$ with $0 < t < \infty$, $u \in \mathbb{C}$ and $|u| \geq 1/R$, where $R > 0$ is any fixed positive number. We have

$$(16) \quad S_n(z; q) = \frac{(-z)^n q^{n^2}}{(q; q)_n} [A_q(q^{-2n}/z) + r_n(z)],$$

where the remainder satisfies

$$(17) \quad |r_n(z)| \leq \left[\frac{q^{n(\delta-1)}}{1-q} + \frac{2}{1-q} \left(\frac{1}{2} \right)^{\lfloor n(2-\delta) \rfloor} \right] A_q(-q^{-2n}/|z|)$$

with $\delta = \min\{\frac{3}{2}, 1 + \frac{t}{4}\}$.

Note that the quantities inside the square brackets in (15) and (17) are exponentially small. Furthermore, the sizes of the q -Airy functions in these two equations for large values of their arguments are about the same as the leading terms in their corresponding approximation formulas in (14) and (16); cf. (29) below.

The investigations mentioned above all started with the explicit expression of $S_n(x; q)$ given in (2). In [14], Wang and Wong used a different method, namely, the Riemann-Hilbert approach, to get a uniform asymptotic expansion of the Stieltjes-Wigert polynomials $p_n(x)$ in terms of Airy functions. But, the main result in [14] needs a correction, and the correction is that the parameter k in (1) and (3) should depend on n and tend to infinity as n tends to infinity. In other words, the result in [14] holds with a varying weight and the number q in (1) is required to approach 1. In fact, more precisely, if $k \sim n^\sigma$ as $n \rightarrow \infty$ and $0 < \sigma < \frac{1}{2}$, then all formulas in [14] remain valid; if $\sigma \geq \frac{1}{2}$, then some equations need be amended, but the

main result still holds; see also Baik and Suidan [1]. Note that the results in Theorem 1 hold even when q tends to 1. Comparing the results in Theorem 1 and in [14], we will establish the following limiting relation between the q -Airy function and the ordinary Airy function:

Theorem 2. *Let $\xi(x)$ denote the function defined by*

$$(18) \quad \frac{2}{3}[\xi(x)]^{3/2} = \frac{1}{\log(1/q)} \int_0^{\log(4x)} \arctan \sqrt{e^s - 1} ds.$$

Then, for fixed $x > 1/4$, the q -Airy function has the following asymptotic approximation

$$(19) \quad A_q(\sqrt{q}x) \sim 2\sqrt{\pi} \exp \left\{ \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\} \left(\frac{\xi(x)}{4x - 1} \right)^{\frac{1}{4}} \text{Ai}(-\xi(x))$$

as $q \rightarrow 1$.

This result in fact holds for any complex x , if we replace $\xi(x)$ in (19) by $\tilde{\xi}(x)$ defined by

$$\frac{2}{3}[-\tilde{\xi}(x)]^{3/2} = \frac{1}{\log q} \left[\int_0^{\log(4x)} \log(1 + \sqrt{1 - e^s}) ds - \frac{\log^2(4x)}{4} \right].$$

In view of this result, A_q is indeed a q -analogue of the Airy function.

2. GLOBAL ASYMPTOTICS OF STIELTJES-WIGERT POLYNOMIALS

Proof of Theorem 1.

If $u = 0$, then $z = uq^{-nt} = 0$ and

$$(q; q)_n S_n(0; q) = A_q(0) = 1$$

by the series representations in (2) and (7). Hence, the result in (14)-(15) follows immediately, and we just need consider the case when $u \neq 0$. Noting that

$$z = uq^{-nt} = \frac{u}{|u|} q^{-n \left(t - \frac{\log |u|}{n \log q} \right)},$$

we may assume $|u| = 1$ without loss of generality.

For notational convenience, we put

$$(20) \quad \begin{aligned} r_n(z) &= (q; q)_n S_n(z; q) - A_q(z) \\ &= \sum_{j=0}^n \left[\frac{(q; q)_n}{(q; q)_{n-j}} - 1 \right] \frac{q^{j^2}}{(q; q)_j} (-z)^j - \sum_{j=n+1}^{\infty} \frac{q^{j^2}}{(q; q)_j} (-z)^j \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{j=0}^{\lfloor n\sigma \rfloor} \left[\frac{(q; q)_n}{(q; q)_{n-j}} - 1 \right] \frac{q^{j^2}}{(q; q)_j} (-z)^j, \\ I_2 &= \sum_{j=\lfloor n\sigma \rfloor + 1}^n \left[\frac{(q; q)_n}{(q; q)_{n-j}} - 1 \right] \frac{q^{j^2}}{(q; q)_j} (-z)^j, \\ I_3 &= \sum_{j=n+1}^{\infty} \frac{q^{j^2}}{(q; q)_j} (-z)^j, \end{aligned}$$

and $0 < \sigma < 1$ is a constant to be specified later. In view of the inequality $1 - ab < (1 - a) + (1 - b)$ for any $a, b \in (0, 1)$, we have

$$(21) \quad 1 - \frac{(q; q)_n}{(q; q)_{n-j}} = 1 - (q^{n-j+1}; q)_j < \sum_{i=1}^j q^{n-j+i} < \frac{q^{n-j}}{1-q}.$$

Thus, we have

$$(22) \quad |I_1| \leq \sum_{j=0}^{\lfloor n\sigma \rfloor} \frac{q^{n-j}}{1-q} \frac{q^{j^2}}{(q; q)_j} |z|^j < \frac{q^{n(1-\sigma)}}{1-q} A_q(-|z|).$$

Let

$$(23) \quad \sigma = \max \left\{ \frac{1}{2}, \frac{t}{4} + \frac{1}{2} \right\}, \quad l = \max \left\{ \frac{n}{4}, \lfloor \frac{nt}{2} \rfloor + 1 \right\}.$$

It is readily seen that $\frac{t}{2} < \sigma < 1$ and that $\frac{nt}{2} < l < n$ for sufficiently large n . For $j \geq \lfloor n\sigma \rfloor$ and n sufficiently large, there exists $\gamma > 0$ such that

$$(j^2 - ntj) - (l^2 - ntl) = (j - l)(j + l - nt) > (j - l)^2 \geq \gamma^2 j^2.$$

Then, it follows that

$$\begin{aligned} |I_2 + I_3| &\leq \frac{1}{1-q} \sum_{j=\lfloor n\sigma \rfloor+1}^{\infty} \frac{q^{j^2}}{(q; q)_j} |z|^j \\ &= \frac{q^{l^2-ntl}}{1-q} \sum_{j=\lfloor n\sigma \rfloor+1}^{\infty} \frac{q^{(j^2-ntj)-(l^2-ntl)}}{(q; q)_j} \\ &< \frac{q^{l^2-ntl}}{(1-q)(q; q)_{\infty}} \sum_{j=\lfloor n\sigma \rfloor+1}^{\infty} q^{\gamma^2 j^2}. \end{aligned}$$

Since $q^{\gamma^2 j} \leq \frac{1}{2}$ for $j \geq \lfloor n\sigma \rfloor$ and sufficiently large n , we have

$$(24) \quad \begin{aligned} |I_2 + I_3| &\leq \frac{q^{l^2-ntl}}{(1-q)(q; q)_{\infty}} \sum_{j=\lfloor n\sigma \rfloor+1}^{\infty} \left(\frac{1}{2} \right)^j \\ &\leq \frac{q^{l^2-ntl}}{(1-q)(q; q)_{\infty}} \left(\frac{1}{2} \right)^{\lfloor n\sigma \rfloor}. \end{aligned}$$

Similar to the inequality in (21), we have

$$1 - \frac{(q; q)_{\infty}}{(q; q)_l} = 1 - (q^{l+1}; q)_{\infty} < \frac{q^l}{1-q} \leq \frac{q^{\frac{n}{4}}}{1-q} < \frac{1}{2}$$

for sufficiently large n , which leads to

$$(25) \quad \frac{(q; q)_l}{(q; q)_{\infty}} < \frac{1}{1 - \frac{1}{2}} < 2.$$

Moreover, we have

$$(26) \quad \frac{q^{l^2-ntl}}{(q; q)_l} \leq \sum_{j=0}^{\infty} \frac{q^{j^2-ntj}}{(q; q)_j} \leq \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q; q)_j} |q^{-nt}u|^j = A_q(-|z|).$$

A combination of the inequalities in (24), (25) and (26) gives

$$|I_2 + I_3| \leq \frac{2}{1-q} \left(\frac{1}{2}\right)^{\lfloor n\sigma \rfloor} A_q(-|z|),$$

which, together with (20) and (22), yields (15).

The result in (16)-(17) can be proved in a similar manner. One can also obtain this directly from (14), (15) and the symmetry relation of $S_n(z; q)$ mentioned in (4).

3. LIMITING BEHAVIOR OF $A_q(z)$ AS $q \rightarrow 1$

It is known that $A_q(z)$ has infinitely many positive zeros and satisfies the three-term recurrence relation [5]

$$(27) \quad A_q(z) - A_q(qz) + qzA_q(q^2z) = 0.$$

Moreover, Zhang [16] has shown that

$$(28) \quad \lim_{q \rightarrow 1^-} A_q((1-q)z) = e^{-z}$$

for any fixed $z \in \mathbb{C}$. In [13, Proposition 1], Wang and Wong proved that

$$(29) \quad A_q(z) = \frac{(-z)^m q^{m^2}}{(q; q)_\infty} [\Theta_q(-q^{2m}z) + \mathcal{O}(q^{m(1-\delta)})]$$

as $z \rightarrow \infty$, where $m := \lfloor \frac{\ln|z|}{-2\ln q} \rfloor$ and $\delta > 0$ is any small number; see also Zhang [16, Theorem 2.1].

In this section, we shall establish the limiting relation of $A_q(z)$, as $q \rightarrow 1$, stated in Theorem 2. Let us first review some of the results given in Wang and Wong [14]. Let

$$(30) \quad k := k(n) = n^{\frac{1}{4}}.$$

Then

$$(31) \quad q = \exp\{-(2k^2)^{-1}\} = \exp\{-(2\sqrt{n})^{-1}\} = 1 - \frac{1}{2\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right)$$

as $n \rightarrow \infty$. In Sec. 1 we have commented that the main result in Wang and Wong [14] holds if the parameter k in (1) depends on n and satisfies a growth condition such as the one given in (30). The MRS numbers α_n and β_n have been calculated in [14], and are explicitly given in equations (3.19) and (3.20). We note that

$$(32) \quad \alpha_n \sim \frac{1}{4}, \quad \beta_n \sim 4q^{-(2n+1)}, \quad \text{as } n \rightarrow \infty;$$

cf. [14, eq. (2.4)]. Furthermore, the change of variable $t \rightarrow y$ defined by

$$(33) \quad y = \sqrt{\alpha_n \beta_n} \exp\left[\frac{t}{2} \log(\beta_n / \alpha_n)\right]$$

takes the interval $-1 \leq t \leq 1$ onto the interval $\alpha_n \leq y \leq \beta_n$, where $y = 1/(xq^{2n+1})$; see [14, (2.14)]. Let $\pi_n(x)$ denote the monic Stieltjes-Wigert polynomial

$$\pi_n(x) = \frac{p_n(x)}{\gamma_n},$$

where $p_n(x)$ is the polynomial given in (5) and

$$\gamma_n = q^{n^2+n+1/4} / \sqrt{(q; q)_n};$$

that is

$$(34) \quad \pi_n(x) = (-1)^n q^{-n^2-n/2} (q; q)_n S_n(q^{1/2}x; q).$$

The major results in [14] is the asymptotic formula

$$(35) \quad \pi_n(y) = \frac{\sqrt{\pi} e^{ln/2}}{\sqrt{w(y)}} \left\{ N^{\frac{1}{6}} \text{Ai} \left(N^{\frac{2}{3}} \eta_n(t) \right) A(y, n) + \mathcal{O} \left(N^{-\frac{1}{6}} \right) \right\},$$

where $N = n + \frac{1}{2}$,

$$(36) \quad A(y, n) = \frac{[\eta_n(t)]^{1/4} (\beta_n - \alpha_n)^{1/2}}{[(y - \alpha_n)(y - \beta_n)]^{1/4}},$$

$$\frac{2}{3} [-\eta_n(t)]^{3/2} = \frac{a}{N \log(1/q)} \int_t^1 \arctan \frac{\sqrt{(e^{a\tau} - e^{-a})(e^a - e^{a\tau})}}{e^{a\tau} + 1} d\tau$$

and

$$a := \frac{1}{2} \log \frac{\beta_n}{\alpha_n};$$

see [14, (2.11), (2.12) and (6.7)]. Recall that y in (35) and (36) is a function of t , given in (33). Using the formula [14, (6.8)]

$$\frac{2}{3} [-\eta_n(t)]^{3/2} = \frac{1}{N \log(1/q)} \int_0^{a(1-t)} \arctan \sqrt{e^s - 1} ds + \mathcal{O} \left(q^{\frac{1}{2}\delta N} \right),$$

where $-1 + \delta < t < 1$, we have

$$(37) \quad N^{\frac{2}{3}} \eta_n \sim -\xi(x)$$

as $n \rightarrow \infty$, where $\xi(x)$ is the function defined in (18). The last equation gives

$$(38) \quad \text{Ai}(N^{2/3} \eta_n) \sim \text{Ai}(-\xi(x)), \quad n \rightarrow \infty.$$

Next, we show that

$$(39) \quad N^{\frac{1}{6}} A(y, n) \sim 2\sqrt{x} \left(\frac{\xi(x)}{4x - 1} \right)^{\frac{1}{4}}, \quad n \rightarrow \infty.$$

To this end, we note from (36) that

$$(40) \quad N^{\frac{1}{6}} A \left(\frac{1}{xq^{2n+1}}, n \right) = \left\{ \frac{N^{2/3} \eta_n(t) (\beta_n - \alpha_n)^2}{[1/xq^{2n+1} - \alpha_n][1/xq^{2n+1} - \beta_n]} \right\}^{\frac{1}{4}}$$

$$= \left\{ \frac{N^{2/3} \eta_n(t) (\beta_n - \alpha_n)^2 x^2 q^{2(2n+1)}}{(1 - \alpha_n x q^{2n+1})(1 - \beta_n x q^{2n+1})} \right\}^{\frac{1}{4}}.$$

By using equations (3.17) and (3.18) in [14], we have

$$(41) \quad \alpha_n \beta_n = q^{-(2n+1)}.$$

Since β_n is large, the last two equations, together with (37), give

$$(42) \quad N^{\frac{1}{6}} A \left(\frac{1}{xq^{2n+1}}, n \right) \sim \left(\frac{-\xi(x)x^2/\alpha_n^2}{1 - x/\alpha_n} \right)^{\frac{1}{4}}, \quad n \rightarrow \infty,$$

thus proving (39). Here, we have also made use of the fact that $\alpha_n \sim \frac{1}{4}$ as $n \rightarrow \infty$.

Finally, we evaluate the asymptotics of

$$(43) \quad \frac{\sqrt{\pi} e^{\frac{1}{2}l_n}}{y^n \sqrt{w(y)}} = \sqrt{\pi} \exp \left\{ \frac{1}{2}l_n + \frac{1}{2}k^2 \log^2 y - \frac{1}{2} \log \frac{k}{\sqrt{\pi}} - n \log y \right\}.$$

Recall $y = 1/xq^{2n+1}$. Using the formula [14, (3.29)]

$$l_n = \frac{N(N-1)}{k^2} - \frac{k^2 \pi^2}{3} + \log \frac{k}{\sqrt{\pi}} + \mathcal{O}(Nq^N),$$

we obtain

$$(44) \quad \frac{\sqrt{\pi} e^{\frac{1}{2}l_n}}{y^n \sqrt{w(y)}} \sim \sqrt{\pi} \exp \left\{ \frac{N(N-1)}{2k^2} - \frac{k^2 \pi^2}{6} + \frac{1}{2}k^2 \log^2(xq^{2n+1}) + n \log(xq^{2n+1}) \right\}.$$

Since $\log(1/q) = 1/2k^2$ and $N = n + \frac{1}{2}$, one can show that the right-hand side of (44) is equal to

$$(45) \quad \sqrt{\pi} \exp \left\{ -\frac{1}{2} \log x + \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\} = \sqrt{\frac{\pi}{x}} \exp \left\{ \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\}.$$

Hence, we obtain

$$(46) \quad \frac{\sqrt{\pi} e^{\frac{1}{2}l_n}}{y^n \sqrt{w(y)}} \sim \sqrt{\frac{\pi}{x}} \exp \left\{ \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\}, \quad n \rightarrow \infty.$$

A combination of (35), (39) and (46) yields

$$(47) \quad (xq^{2n+1})^n \pi_n(1/xq^{2n+1}) \sim \sqrt{\frac{\pi}{x}} \exp \left\{ \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\} 2\sqrt{x} \left(\frac{\xi(x)}{4x-1} \right)^{\frac{1}{4}} \text{Ai}(-\xi(x)).$$

Therefore, we have

$$(xq^{2n+1})^n \pi_n(1/xq^{2n+1}) = (q; q)_n (-q^{1/2} x q^n)^n S_n(q^{-2n}/q^{1/2} x; q) = (q; q)_n S_n(q^{\frac{1}{2}} x; q).$$

Here, we have made use of the symmetry relation of $S_n(z; q)$ given in (4). Note that the result in Theorem 1 in fact holds with q satisfying (31). Thus, Theorem 1 gives

$$(48) \quad (xq^{2n+1})^n \pi_n(1/xq^{2n+1}) = A_q(\sqrt{q}x) + r_n(\sqrt{q}x).$$

Coupling (47) and (48), we obtain the desired formula

$$A_q(\sqrt{q}x) \sim 2\sqrt{\pi} \exp \left\{ \frac{3 \log^2 x - \pi^2}{12 \log(1/q)} \right\} \left(\frac{\xi(x)}{4x-1} \right)^{\frac{1}{4}} \text{Ai}(-\xi(x))$$

as $q \rightarrow 1^-$.

4. NUMERICAL VERIFICATION

In the following tables, we have used a Maple-aided program to verify the asymptotic formulas in Theorem 1 and the limiting relation in Theorem 2. The values are represented in scientific notations; for example,

$$-2.325\text{e-}3 = -2.325 \times 10^{-3} = -0.00235.$$

In Table 1, ‘True’ stands for the true value of $(q; q)_n S_n(uq^{-nt}; q)$ by summing the series in (2); ‘Approx.’ stands for the approximate value obtained from (14); ‘Error’ is the relative

error of the approximate value. The degree of the polynomial is $n = 50$ and $q = 0.5$ in Table 1, u takes the values of 1, -1 and $1+i$, and t takes the values of 0, 0.5, 1.0, 1.2 and 1.6. Here, we would like to mention that the approximate values are very close to the true values. For instance, in the case when $u = t = 1$, the true value is $-5.83981318477869 \cdots \times 10^{187}$, whereas the approximate value is $-5.83981318477868 \cdots \times 10^{187}$. If we take only a few digits as what we have done in Table 1, the two values all appear to be nearly the same. In Table 2, ‘True’ stands for the true value of $A_q(\sqrt{q}x)$ by summing the series in (7); ‘Approx.’ stands for the approximate value obtained from the quantity on the right-hand side of (19); ‘Error’ is the relative error of the approximate value. We examine the cases when $x = 0.5, 1.0, 4.0, 10, 20$ and $q = 0.9, 0.92, 0.94, 0.96, 0.98, 0.99$.

TABLE 1. Numerical Verification of Theorem 1

$\begin{smallmatrix} t \\ u \end{smallmatrix}$		0	0.5	0.8	1.0	1.2	1.6
1	True	0.16076	-9.3534e42	1.0831e120	-5.8398e187	3.5453e270	1.8649e481
	Approx.	0.16076	-9.3534e42	1.0831e120	-5.8398e187	3.5453e270	1.8649e481
	Error	2.99e-15	2.98e-8	1.78e-15	1.18e-15	6.05e-13	6.36e-7
-1	True	2.17267	8.0063e47	1.9036e121	1.0264e189	6.2313e271	3.2740e482
	Approx.	2.17267	8.0063e47	1.9306e121	1.0264e189	6.2312e271	3.2778e482
	Error	6.31e-16	6.12e-12	1.11e-9	3.54e-8	1.13e-6	1.16e-3

$\begin{smallmatrix} t \\ u \end{smallmatrix}$		0	0.5	1.0	1.6
1+i	True	0.0117-0.6786i	(1.92+8.38i)e48	(-8.18-1.87i)e191	(4.107-2.571i)e487
	Approx.	0.0117-0.6786i	(1.92+8.38i)e48	(-8.18-1.87i)e191	(4.106-2.578i)e487
	Error	8.83e-17	3.16e-12	1.39e-8	4.55e-4

TABLE 2. Numerical Verification of Theorem 2

$\begin{smallmatrix} q \\ x \end{smallmatrix}$		0.9	0.92	0.94	0.96	0.98	0.99
0.5	True	-2.325e-3	-3.826e-4	1.120e-5	-2.966e-8	5.080e-16	4.9298e-32
	Approx.	-2.320e-3	-3.819e-4	1.118e-5	-2.964e-8	5.078e-16	4.9303e-32
	Error	0.0022	0.0018	0.0012	0.00080	0.00038	0.0000995
1.0	True	-5.171e-4	2.978e-5	-2.556e-6	1.326e-9	2.178e-18	4.1417e-36
	Approx.	-5.159e-4	2.973e-5	-2.553e-6	1.325e-9	2.177e-18	4.1408e-36
	Error	0.0022	0.0018	0.0013	0.00087	0.00043	0.00021
4.0	True	0.034973	0.01680	4.4202e-4	-1.0084e-4	4.4912e-8	-5.6869e-16
	Approx.	0.034891	0.01677	4.1898e-4	-1.0073e-4	4.4893e-8	-5.6853e-16
	Error	0.00235	0.0018	0.0028	0.00107	0.00043	0.00028
10	True	38.6522	-247.876	2715.83	43744.8	3.3978e10	2.1941e21
	Approx.	38.5316	-247.372	2712.29	43745.2	3.3961e10	2.1944e21
	Error	0.0031	0.0020	0.0013	0.00022	0.00051	0.00014
20	True	-2.0951e5	-3.5927e6	-5.9716e9	7.3472e14	-6.1129e29	1.5900e61
	Approx.	-2.0884e5	-3.5801e6	-5.9645e9	7.3418e14	-6.1124e29	1.5897e61
	Error	0.00320	0.00349	0.00119	0.00073	0.00007	0.00023

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